On the Angular Dependence of the Radiative Gluon Spectrum

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The induced momentum spectrum of soft gluons radiated from a high energy quark produced in and propagating through a QCD medium is reexamined in the BDMPS formalism. A mistake in our published work (Physical Review C60 (1999) 064902) is corrected. The correct dependence of the fractional induced loss $R(\theta_{\rm cone})$ as a universal function of the variable $\theta^2_{\rm cone} L^3 \hat{q}$ where L is the size of the medium and $\hat{q}$ the transport coefficient is presented. We add the proof that the radiated gluon momentum spectrum derived in our formalism is equivalent with the one derived in the Zakharov-Wiedemann approach.

I. INTRODUCTION

In the present paper we correct an error made in the calculation for the angular distribution of radiated gluons as presented in [1]. We recalculate the quantitative predictions of the integrated energy loss outside an angular cone, defining the jet, with fixed opening angle, $\Delta E(\theta_{\rm cone})$, and derive the expression for the ratio $R(\theta_{\rm cone}) = \Delta E(\theta_{\rm cone})/\Delta E$, where ΔE is the completely integrated loss. We confirm our previous result that $R(\theta_{\rm cone})$ is a universal function of the variable $\theta^2_{\rm cone}\hat{q}L^3$, where $\hat{q}$ is the transport coefficient characteristic of the medium. In particular we now find that $R(\theta_{\rm cone})$ is strongly peaked at small values of $\theta_{\rm cone}$ ($\theta^2_{\rm cone}\hat{q}L^3 \simeq 1/10$), a feature already obtained by Wiedemann [2]. Finally, we explicitly show that the induced radiated gluon momentum spectrum derived in our formalism is equivalent with the one derived in the Zakharov-Wiedemann approach [3], namely in the form given in [2].

II. MOMENTUM SPECTRUM OF RADIATED GLUONS

In this section we repeat shortly the discussion of the induced momentum spectrum of soft gluon emission ($x \to 0$) from a fast quark jet, which is produced *inside* the medium by a hard scattering at time $t = 0$. From the production point the quark propagates over a length L of QCD matter, carrying out many scatterings via gluon exchange with the medium.

In the BDMPS approach [4] the spectrum per unit length z of the medium consists of two terms: one corresponding to an "on-shell" quark, as if that quark were entering the medium, and one corresponding to a hard production vertex. From [1] we quote the induced gluon spectrum in the soft ω limit:

$$\frac{\omega dI}{d\omega dz d^2\underline{U}} = \frac{\alpha_s C_F}{\pi^2 L} 2\,Re \int_0^L dt_2 \int d^2\underline{Q} \left\{ \int_0^{t_2} dt_1 \rho\sigma \frac{N_c}{2C_F} f(\underline{U}+\underline{Q}, t_2 - t_1) + f_h(\underline{U}+\underline{Q}, t_2) \right\}$$

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$$\times\, \rho\sigma \frac{N_c}{2C_F} 2 \left[\frac{\underline{U}+\underline{Q}}{(\underline{U}+\underline{Q})^2} - \frac{\underline{U}}{\underline{U}^2} \right] V(\underline{Q})\, \mathcal{F}_{fsi} \Bigg|_{\tilde{\kappa}}^{\tilde{\kappa}=0} , \tag{1}$$

where f_h differs from f in that the gluon emission time has been evaluated at the time of the hard interaction, $t = 0$. We remind that the the medium-independent factorisation contribution is eliminated by a subtraction of the value of the integrals at $\tilde{\kappa} = 0$, where this parameter is

$$\tilde{\kappa} = \frac{2C_F}{N_c} \frac{\lambda \mu^2}{2\omega}. \tag{2}$$

It depends on medium properties, in particular on the quark mean free path $\lambda = 1/\rho\sigma$, as well as on the gluon energy ω. The other symbols are explained in [1].

Next we convert (1) to impact parameter space as explained in [1], and we rescale the time variable $t = \frac{2C_F}{N_c}\lambda\tau$ and define $\tau_0 = N_c L/2C_F\lambda$. We obtain the spectrum in terms of impact parameter integrals,

$$\frac{\omega dI}{d\omega dz d^2\underline{U}} = \frac{\alpha_s C_F}{\pi^2 L} 2Re \int_0^{\tau_0} d\tau_2 \int \frac{d^2B_1}{(2\pi)^2} \frac{d^2B_2}{(2\pi)^2}\, e^{i(\underline{B}_1 - \underline{B}_2)\cdot \underline{U}} \left\{ \int_0^{\tau_2} d\tau_1 \tilde{f}(\underline{B}_1, \tau_2 - \tau_1) + \tilde{f}_h(\underline{B}_1, \tau_2) \right\}$$
$$\times\, \frac{4\pi i \underline{B}_2}{\underline{B}_2^2} \left[\tilde{V}(\underline{B}_1 - \underline{B}_2) - \tilde{V}(\underline{B}_1) \right] e^{-\frac{\tilde{v}}{2}(\underline{B}_1 - \underline{B}_2)^2(\tau_0 - \tau_2)} \Bigg|_{\tilde{\kappa}}^{\tilde{\kappa}=0} . \tag{3}$$

We still have to evaluate the explicit expressions for $\tilde{f}$, and especially for $\tilde{f}_h$. We recall that the amplitude $\tilde{f}(\underline{B}, \tau)$ is given by

$$\tilde{f}(\underline{B}, \tau) = -\frac{i\pi\tilde{v}}{\cos^2 \omega_0 \tau} \underline{B} \exp\left(-\frac{i}{2} m\omega_0 \underline{B}^2 \tan \omega_0 \tau \right) , \tag{4}$$

with $m = -1/2\tilde{\kappa}$ and $\omega_0 = \sqrt{2i\tilde{\kappa}\tilde{v}}$. From the fact that $\tilde{f}_h$ and $\tilde{f}$ have the same time evolution given by the harmonic oscillator Green function $G(\underline{B}_2, \tau_2; \underline{B}_1, \tau_1)$ (cf. Eq.(5.6) of [4]), i.e. in Eq. (3),

$$\tilde{f}_h(\underline{B}_1, \tau_2) \;=\; \int d^2B\, G(\underline{B}_1, \tau_2; \underline{B}, \tau = 0)\, \tilde{f}_h(\underline{B}, \tau = 0) , \tag{5}$$

while the initial conditions satisfy (in the soft gluon limit) $\tilde{f}_h(\underline{B}, 0) = \frac{2}{\tilde{v}\underline{B}^2} \tilde{f}(\underline{B}, 0) = -2\pi i \underline{B}/\underline{B}^2$ as can be seen from Eqs.(17b) and (37) of [4].

Here we note the mistake in our paper [1]: instead of the correct Eq.(5) we have taken

$$\tilde{f}_h(\underline{B}_1, \tau) = \frac{2}{\tilde{v}\underline{B}_1^2} \tilde{f}(\underline{B}_1, \tau) = \frac{2}{\tilde{v}\underline{B}_1^2} \int d^2B\, G(\underline{B}_1, \tau; \underline{B}, \tau = 0)\, \tilde{f}(\underline{B}, \tau = 0) , \tag{6}$$

i.e. effectively we have taken out of the integrand in Eq.(6) the factor $1/\underline{B}^2$, and replaced it by $1/\underline{B}_1^2$!

Eq.(5) allows to derive the explicit time and impact parameter dependence of $\tilde{f}_h$, which reads,

$$\tilde{f}_h(\underline{B}, \tau) = -2\pi i \underline{B}/\underline{B}^2 \left[\exp\left(-\frac{i}{2} m\omega_0 \underline{B}^2 \tan \omega_0 \tau \right) - \exp\left(\frac{i}{2} m\omega_0 \frac{\underline{B}^2}{\tan \omega_0 \tau} \right) \right] . \tag{7}$$

Based on this correct treatment of the hard production vertex the soft gluon radiation spectrum can be rewritten as

$$\frac{\omega dI}{d\omega dz d^2\underline{U}} = -4 \frac{\alpha_s C_F \tilde{v}}{L} Re \int_0^{\tau_0} d\tau \int \frac{d^2B_1}{(2\pi)^2} \frac{d^2B_2}{(2\pi)^2}\, e^{i(\underline{B}_1 - \underline{B}_2)\cdot \underline{U}} \exp\left(\frac{i}{2} m\omega_0 \frac{\underline{B}_1^2}{\tan \omega_0 \tau} \right)$$
$$\times\, \frac{\underline{B}_1 \cdot \underline{B}_2}{\underline{B}_1^2 \underline{B}_2^2} \left[(\underline{B}_1 - \underline{B}_2)^2 - \underline{B}_1^2 \right] e^{-\frac{\tilde{v}}{2}(\underline{B}_1 - \underline{B}_2)^2(\tau_0 - \tau)} \Bigg|_{\tilde{\kappa}}^{\tilde{\kappa}=0} . \tag{8}$$

The $\underline{\mathrm{B}}$-space integral is performed as in [1] and expressed in terms of the function

$$J(\underline{U}, \alpha, \beta) = \frac{1}{16\pi^2} \frac{1}{\alpha(\alpha + \beta)} e^{-\frac{\underline{U}^2}{4(\alpha+\beta)}} . \tag{9}$$

III. INDUCED RADIATIVE ENERGY LOSS OF A HARD QUARK JET IN A FINITE CONE

In the following we recalculate the integrated loss *outside* an angular cone of opening angle $\theta_{\rm cone}$,

$$\Delta E(\theta_{\rm cone}) = L \int_0^\infty d\omega \int_{\theta_{\rm cone}}^{\pi} \frac{\omega dI}{d\omega dz d\theta} d\theta. \tag{10}$$

We note that for $\theta_{\rm cone} = 0$ the total loss is obtained, namely [4]

$$\Delta E = \frac{\alpha_s N_c}{4} \hat{q} L^2. \tag{11}$$

We consider the normalized loss [1] by defining the ratio

$$R(\theta_{\rm cone}) = \frac{\Delta E(\theta_{\rm cone})}{\Delta E}, \tag{12}$$

with $R(\theta_{\rm cone} = 0) = 1.$, by using the same (dimensionless) variables and definitions as in [1].

We confirm that the ratio $R(\theta_{\rm cone})$ turns out to depend on one single dimensionless variable

$$R = R(c(L)\theta_{\rm cone}), \tag{13}$$

where

$$c^2(L) = \frac{N_c}{2C_F} \hat{q} \, (L/2)^3 \, . \tag{14}$$

This “scaling behaviour” of R means that the medium and size dependence is universally contained in the function $c(L)$, which is a function of the transport coefficient $\hat{q} = \tilde{v}\mu^2/\lambda$ and of the length L, as defined by (14).

As a consequence the discussion of the medium properties is qualitatively the same as in [1], and does not need to be repeated here. Quantitatively the corrected ratio R is plotted in Fig. 1.

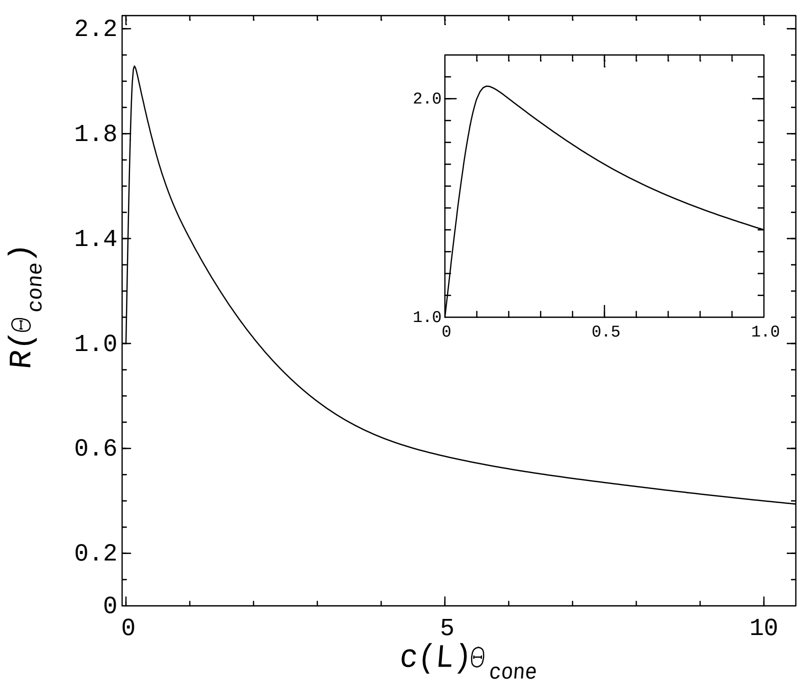

FIG. 1. Fractional induced loss $R(\theta_{\rm cone})$ as a function of $c(L)\theta_{\rm cone}$.

A sharp peak at small values of $c(L)\theta_{\rm cone} \simeq 0.1 - 0.2$, where $R(\theta_{\rm cone}) \simeq 2.$, can be seen in this figure, especially with the help of the insert. This behaviour of the loss $\Delta E(\theta_{\rm cone})$ has been already noticed by Wiedemann in [2]. For $c(L)\theta_{\rm cone} > 10$ the dependence is numerically as given in our previous work [1].

When comparing hot and cold QCD matter we recall [1] that for fixed L

$$c(L)\Big|_{\rm HOT} \gg c(L)\big|_{\rm COLD}. \tag{15}$$

This implies that e.g. for $L = 5fm$, the position of the peak seen in Fig. 1 corresponds to very small cones: $\theta_{\rm cone}^{peak} \leq 2^\circ$ for nuclear, and $\theta_{\rm cone}^{peak} \leq 0.5^\circ$ for hot (at reference temperature $T = 250MeV$) matter, respectively. For $\theta_{\rm cone} > \theta_{\rm cone}^{peak}$ the energy loss, $R(\theta_{\rm cone})$, drops quickly with increasing $\theta_{\rm cone}$, the behaviour already known from the discussion in [1].

For a quark jet produced in the medium the dependence of R on $c(L)\theta_{\rm cone}$ is given for further reference, expressing explicitly the property Eq. (13) by

$$\begin{aligned} R(\theta_{\rm cone}) = \frac{4}{\pi}\, Re \int_0^\infty \frac{dx}{x^3} \int_0^1 y^2 dy \int_0^\infty dz \\ \times \frac{1}{\left[z - \frac{y}{(1+i)x\,\tan(1+i)x}\right]^2} \exp\left[-\frac{c^2(L)\theta_{\rm cone}^2}{1+z-y-\frac{y}{(1+i)x\,\tan(1+i)x}} \left(\frac{y}{x}\right)^4\right]. \end{aligned} \tag{16}$$

The subtraction term in (16) with $\tilde{\kappa} = 0$ vanishes.

For the case of a quark produced outside the medium the calculation is the same as in [1].

IV. EQUIVALENCE OF THE TWO FORMALISMS

We now show that Eq. (3) can be expressed in a form identical to Eq. (2.1), respectively Eqs. (A.5) and (A.6) of [2]. This derivation of the equivalence of our formalism with the one by Wiedemann [2] and Zakharov [3] generalizes the proof given in [4], but here for the full gluon momentum spectrum Eq. (1).

As in [4] we express $\tilde{f}(\underline{B}_1, \tau_2 - \tau_1)$ and $\tilde{f}_h(\underline{B}_1, \tau_2)$ in terms of the Green function G, as given by Eq. (5). The function G obeys the same differential equation as given for $\tilde{f}$ [4],

$$\frac{\partial}{\partial \tau_1} G(\underline{B}_1, \tau_2; \underline{B}, \tau_1) \;=\; i\tilde{\kappa}\, \nabla_{\underline{B}}^2 G + \frac{\tilde{v}}{2} \underline{B}^2\, G\,. \tag{17}$$

Using this Eq. (17) we substitute the combination of $\partial G/\partial\tau_1$ and $\nabla_{\underline{B}}^2 G$ for this Green function G which expresses the τ dependence of $\tilde{f}(\underline{B}_1, \tau_2 - \tau_1)$ in Eq. (3). The lower limit of the τ_1-integral of $\partial G/\partial\tau_1$ exactly cancels the term coming from $\tilde{f}_h(\underline{B}_1, \tau_2)$, while its upper limit, at τ_2, contains

$$G(\underline{B}_1, \tau_2; \underline{B}, \tau_2) \;=\; \delta^2\left(\underline{B}_1 - \underline{B}\right),$$

which is $\tilde{\kappa}$-independent and therefore canceled by the $\tilde{\kappa}\!=\!0$ subtraction term present in Eq. (3). Thus, we arrive at

$$\begin{aligned} \frac{\omega dI}{d\omega dz d^2\underline{U}} = 4\frac{\alpha_s C_F}{L}\, \tilde{\kappa}\tilde{v} Re(-i) \int_0^{\tau_0} d\tau_2 \int_0^{\tau_2} d\tau_1 \int \frac{d^2 B_1}{(2\pi)^2} \frac{d^2 B_2}{(2\pi)^2} \int d^2 B\, e^{i(\underline{B}_1 - \underline{B}_2)\cdot \underline{U}} \\ \times \frac{\underline{B}\cdot\underline{B}_2}{\underline{B}^2 \underline{B}_2^2} \left[(\underline{B}_1 - \underline{B}_2)^2 - \underline{B}_1^2\right] e^{-\frac{\tilde{v}}{2}(\underline{B}_1 - \underline{B}_2)^2(\tau_0 - \tau_2)}\, \nabla_{\underline{B}}^2 G(\underline{B}_1, \tau_2; \underline{B}, \tau_1). \end{aligned} \tag{18}$$

Using

$$\nabla_{\underline{B}} \cdot \frac{\underline{B}}{\underline{B}^2} \;=\; 2\pi\delta^2(\underline{B})\,, \tag{19}$$

and after integrating one $\nabla_{\underline{B}}$ derivative in Eq. (18) by parts one finds

$$\int d^2 B\, \frac{\underline{B}}{\underline{B}^2}\, \nabla_{\underline{B}}^2\, G(\underline{B}_1, \tau_2; \underline{B}, \tau_1) \;=\; -2\pi\, \nabla_{\underline{B}}\, G(\underline{B}_1, \tau_2; \underline{B}, \tau_1). \tag{20}$$

Now, expressing

$$\frac{2}{\tilde{v}}(\underline{B}_1 - \underline{B}_2)^2 = \frac{\partial}{\partial\tau_2} e^{-\frac{\tilde{v}}{2}(\underline{B}_1-\underline{B}_2)^2(\tau_0-\tau_2)}, \tag{21}$$

and interchanging integrations in Eq. (18)

$$\int_0^{\tau_0} d\tau_2 \int_0^{\tau_2} d\tau_1 = \int_0^{\tau_0} d\tau_1 \int_{\tau_1}^{\tau_0} d\tau_2\,, \tag{22}$$

a partial integration with respect to τ_2 is performed. Together with the following equation (cf. Eq. (17))

$$\left(\frac{\partial}{\partial\tau_2} + \frac{\tilde{v}}{2}\underline{B}_1^2\right) G(\underline{B}_1, \tau_2; \underline{B}, \tau_1) \;=\; -i\tilde{\kappa}\, \nabla^2_{\underline{B}_1} G(\underline{B}_1, \tau_2; \underline{B}, \tau_1), \tag{23}$$

and a further integration by parts in $\underline{B}_1$ similar to Eq. (20), the spectrum (3) is finally expressed by

$$\begin{aligned}\frac{\omega dI}{d\omega dz d^2\underline{U}} &= 2\frac{\alpha_s C_F}{\pi^2 L}\,\tilde{\kappa}^2 Re \int_0^{\tau_0} d\tau_1 \int_{\tau_1}^{\tau_0} d\tau_2 \int d^2B_1 e^{i\underline{B}_1\cdot\underline{U}}\, e^{-\frac{\tilde{v}}{2}\underline{B}_1^2(\tau_0-\tau_2)}\, \nabla_{\underline{B}_1}\cdot\nabla_{\underline{B}} G(\underline{B}_1, \tau_2; \underline{B}=0, \tau_1) \\ &+ \frac{\alpha_s C_F}{\pi^2 L}\,\tilde{\kappa} Re \int_0^{\tau_0} d\tau_1 \int d^2B_1 e^{i\underline{B}_1\cdot\underline{U}}\,\frac{\underline{U}}{\underline{U}^2}\cdot\nabla_{\underline{B}} G(\underline{B}_1, \tau_0; \underline{B}=0, \tau_1).\end{aligned} \tag{24}$$

In the second term the transform

$$\int d^2B_2\, e^{-i\underline{B}_2\cdot\underline{U}}\,\frac{\underline{B}_2}{\underline{B}_2^2} = -2\pi i\frac{\underline{U}}{\underline{U}^2} \tag{25}$$

has been inserted. The subtraction terms at $\tilde{\kappa}=0$ vanish.

The two terms on the right hand side of (24) correspond to and are identical with the terms given by Eqs. (A.5) and (A.6), respectively, in the appendix of the paper by Wiedemann [2]. These terms express the gluon radiation spectrum off a quark produced in and propagating through the medium of size L.

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